Topic 6 –

Coordinate systems in $\mathbb{R}^n$

The set $\mathbb{R}^n$ is an example of a mathematical object called a <u>vector space</u>.

The scalars/numbers $\mathbb{R}$ are an example of a mathematical object called a <u>field</u>.

Def:

Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set of r vectors from $\mathbb{R}^n$.

- We say that a vector $\vec{v}$ in $\mathbb{R}^n$ is in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ (or we say in the span of β) if we can write

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_r\vec{v}_r$$

ie you can "make" $\vec{v}$ by scaling and adding the vectors from β

where $c_1, c_2, \ldots, c_r$ are real numbers.

- The expression

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_r\vec{v}_r$$

is called a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$.

- If $r=1$, ie if $\beta = \{\vec{v}_1\}$ and we only have one vector, then we have two cases:

- If $\vec{v}_1 = \vec{0}$, then we say that β is a linearly dependent set of vectors (or just say $\vec{v}_1$ is linearly dependent)
- If $\vec{v}_1 \neq 0$, then we say that β is a linearly independent set of vectors (or say $\vec{v}_1$ is linearly independent).

- If $r \geq 2$, ie β has more than one vector, then we say that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are linearly dependent if one of the vectors is in the span of the other vectors ie if one of the vectors is a linear combination of the others.
If this is not the case, then the vectors are called linearly independent.

<u>Ex:</u> Let $\vec{a} = \langle 0,0 \rangle$ in the vector space $\mathbb{R}^2$.

Let $\beta = \{\vec{a}\} = \{\vec{0}\}$

Then β is a linearly dependent set of vectors.

What is in the span of $\vec{a}$?

If c_1 is a real number, then

$$c_1 \cdot \vec{a} = c_1 \cdot \vec{0} = \vec{0}$$

So the only vector in the span of $\vec{a}$ is itself $\vec{0}$.

Ex: Let $\vec{v} = \langle 1, 2 \rangle$ in the vector space $\mathbb{R}^2$

Let $\beta = \{ \vec{v} \}$.

Since β has one non-zero vector $\vec{v} \neq \vec{0}$ we say that β is a linearly independent set of vectors.

What are some vectors in the span of β?

$$2 \cdot \vec{v} = 2 \langle 1, 2 \rangle = \langle 2, 4 \rangle$$

$$-5 \cdot \vec{v} = -5 \langle 1, 2 \rangle = \langle -5, -10 \rangle$$

$$\pi \cdot \vec{v} = \pi \langle 1, 2 \rangle = \langle \pi, 2\pi \rangle$$

$$-\frac{3}{2} \vec{v} = -\frac{3}{2} \langle 1, 2 \rangle = \langle -\frac{3}{2}, -3 \rangle$$

Some vectors in the span of $\vec{v}$ (or span of β) are:

$\langle 2, 4 \rangle, \langle -5, -10 \rangle, \langle \pi, 2\pi \rangle, \langle -\frac{3}{2}, -3 \rangle$

Any multiple of $\vec{v}$ is in the span of $\vec{v}$.

Ex: Consider the vector space $\mathbb{R}^2$.

Let $\vec{v}_1 = \langle 1, 1 \rangle$, $\vec{v}_2 = \langle 2, 2 \rangle$.

Let $\beta = \{ \vec{v}_1, \vec{v}_2 \}$.

Q: What are some vectors in the span of β?

$$2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = 2\langle 1,1 \rangle + 1 \cdot \langle 2,2 \rangle = \langle 4,4 \rangle$$

$$4 \cdot \vec{v}_1 - 2 \cdot \vec{v}_2 = 4\langle 1,1 \rangle - 2\langle 2,2 \rangle = \langle 0,0 \rangle$$

So, $\langle 3,3 \rangle, \langle 0,0 \rangle$ are in the span of $\beta = \{ \vec{v}_1, \vec{v}_2 \}$.

Note that in general a vector in the span of β is of the form:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \langle 1,1 \rangle + c_2 \langle 2,2 \rangle$$

$$= c_1 \langle 1,1 \rangle + 2c_2 \langle 1,1 \rangle$$

$$= (c_1 + 2c_2) \langle 1,1 \rangle$$

Ie the vectors in the span of $\beta = \{ \vec{v}_1, \vec{v}_2 \}$ are just the vectors in the span of $\vec{v}_1$!

Q: Are the vectors in $\beta = \{\vec{v}_1, \vec{v}_2\}$ linearly dependent or linearly independent?

Note that

$$\langle 2, 2\rangle = 2 \cdot \langle 1, 1\rangle$$

ie

$$\vec{v}_2 = 2\vec{v}_1$$

Thus, since $\vec{v}_2$ is in the span of $\vec{v}_1$ the vectors in β are linearly dependent. This is why the span of $\beta = \{\vec{v}_1, \vec{v}_2\}$ collapsed to just the span of $\vec{v}_1$.

Note: $\vec{v}_2 = 2\vec{v}_1$ can be written $2\vec{v}_1 - 1 \cdot \vec{v}_2 = \vec{0}$

We will use this idea when we give another way to check lin. dep. / lin. ind.

Ex: Consider the vector space $\mathbb{R}^2$.

Let $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$.

Let $\beta = \{\vec{i}, \vec{j}\}$.

Q: What are some vectors in the span of $\vec{i}, \vec{j}$?

$$3 \cdot \vec{i} + 2 \cdot \vec{j} = 3\langle 1,0 \rangle + 2\langle 0,1 \rangle = \langle 3,2 \rangle$$

$$-\vec{i} + \frac{1}{2}\vec{j} = -\langle 1,0 \rangle + \frac{1}{2}\langle 0,1 \rangle = \langle -1, \tfrac{1}{2} \rangle$$

So, $\langle 3,2 \rangle, \langle -1, \frac{1}{2} \rangle$ are in the span of β, ie the span of $\vec{i}, \vec{j}$.

In fact any vector $\vec{v} = \langle a,b \rangle$ is in the span of β because

$$\begin{aligned} \vec{v} = \langle a,b \rangle &= \langle a,0 \rangle + \langle 0,b \rangle \\ &= a\langle 1,0 \rangle + b\langle 0,1 \rangle \\ &= a\vec{i} + b\vec{j} \end{aligned}$$

Thus, the span of $\vec{i}, \vec{j}$ consists of all vectors in $\mathbb{R}^2$.

Q: Is β a linearly independent set?

Is one of the vectors in the span of the other vector?

Let's see.

Can we write $\vec{i} = c\vec{j}$ where c is a scalar?

We would need $\langle 1,0\rangle = c\langle 0,1\rangle$

or $\langle 1,0\rangle = \langle 0,c\rangle$.

This would require $1=0$!

So it's not possible.

Can we write $\vec{j} = c\vec{i}$?

That would require $\langle 0,1\rangle = c\langle 1,0\rangle$

which would need $\langle 0,1\rangle = \langle c,0\rangle$.

This would require $1=0$ which can't happen.

From above we see that $\beta = \{\vec{i}, \vec{j}\}$ is a linearly independent set.

Ex: Consider the vector space $\mathbb{R}^3$.

Let $\vec{v}_1 = \langle -1, 0, 1 \rangle$, $\vec{v}_2 = \langle -5, -3, 2 \rangle$, $\vec{v}_3 = \langle 1, 1, 0 \rangle$.

Then

$$\underbrace{\langle -5, -3, 2 \rangle = 2 \cdot \langle -1, 0, 1 \rangle - 3 \cdot \langle 1, 1, 0 \rangle}_{\vec{v}_2 = 2\vec{v}_1 - 3\vec{v}_3}$$

So, $\vec{v}_2$ is in the span of $\vec{v}_1$ and $\vec{v}_3$.

The above tells us that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent. This is because you can make $\vec{v}_2$ from scaling/adding $\vec{v}_1$ and $\vec{v}_3$.

Note the above equation can be written as:

$$2\vec{v}_1 - \vec{v}_2 - 3\vec{v}_3 = \vec{0}$$

a linear equation relating $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which shows their linear dependence

Here's another way to check for linear independence or linear dependence

Theorem: Consider the vector space $\mathbb{R}^n$.
Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$

β is a linearly independent set if and only if the only solution to the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0}$$

is $c_1 = 0, c_2 = 0, \ldots, c_r = 0$.

If the equation has more solutions then the vectors are linearly dependent.

Ex: Let $\beta = \{\vec{i}, \vec{j}\}$ in $\mathbb{R}^2$.

Let's use this theorem to verify again that β is a linearly independent set of vectors.

We want to solve

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

for c_1, c_2.

We get $c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$

This becomes $\langle c_1, 0 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$

which is $\langle c_1, c_2 \rangle = \langle 0, 0 \rangle$.

Thus, $c_1 = 0, c_2 = 0$.

Since the only solution to

$$c_1 \cdot \vec{i} + c_2 \cdot \vec{j} = \vec{0}$$

is $c_1 = 0, c_2 = 0$ we have that $\beta = \{\vec{i}, \vec{j}\}$ is a linearly independent set of vectors

Ex: Consider the vector space $\mathbb{R}^2$.

Let $\vec{v} = \langle 1, 0 \rangle$, $\vec{w} = \langle -1, 1 \rangle$.

Let $\beta = \{\vec{v}, \vec{w}\}$.

Is β a lin. ind. or lin. dep. set of vectors?

We need to solve

$$c_1 \vec{v} + c_2 \vec{w} = \vec{0}$$

for c_1, c_2.

We get

$$c_1 \langle 1, 0 \rangle + c_2 \langle -1, 1 \rangle = \langle 0, 0 \rangle$$

$$\langle c_1 - c_2, c_2 \rangle = \langle 0, 0 \rangle$$

which gives

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_2 &= 0 \end{aligned}}$$

The only solution to this system is $c_1 = 0$, $c_2 = 0$.

Thus, $\vec{v} = \langle 1, 0 \rangle$, $\vec{w} = \langle -1, 1 \rangle$ are linearly independent.

Ex: Let $\vec{v}_1 = \langle 1,-2,1\rangle$, $\vec{v}_2 = \langle 1,0,1\rangle$, $\vec{v}_3 = \langle 0,1,0\rangle$.

Are these vectors linearly independent or linearly dependent?

We must solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \quad (*)$$

for c_1, c_2, c_3.

We get

$$c_1\langle 1,-2,1\rangle + c_2\langle 1,0,1\rangle + c_3\langle 0,1,0\rangle = \langle 0,0,0\rangle$$

$$\langle c_1+c_2, -2c_1+c_3, c_1+c_2\rangle = \langle 0,0,0\rangle$$

This gives the system

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \\ -2c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

Solving gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[-R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 \qquad &= 0 \\ c_2 + \tfrac{1}{2} c_3 &= 0 \\ 0 &= 0 \end{aligned}}$$

(1)
(2)

We get

$$\boxed{\begin{aligned} c_1 &= -c_2 \\ c_2 &= -\tfrac{1}{2} c_3 \\ c_3 &= t \end{aligned}}$$

(1)
(2)
(3)

So,

$c_3 = t$

$c_2 = -\frac{1}{2} t$

$c_1 = \frac{1}{2} t$

Thus plugging this back into (*) above we get

$$\frac{1}{2} t \vec{v}_1 - \frac{1}{2} t \vec{v}_2 + t \vec{v}_3 = \vec{0}$$

for any t.

For example if $t=2$ we get

$$\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 = \vec{0}$$

Thus, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

For example,

$$\vec{v}_1 = \vec{v}_2 - 2\vec{v}_3$$

or

$$\vec{v}_2 = \vec{v}_1 + 2\vec{v}_3 .$$

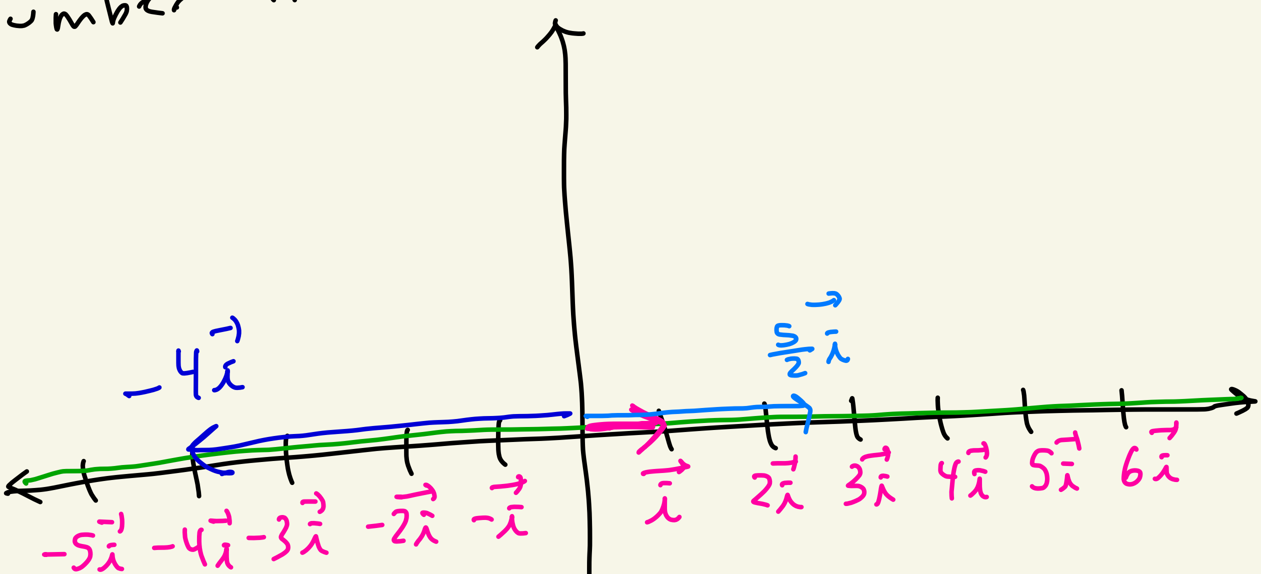
The idea here is that you have some redundancies, ie you can make $\vec{v}_1$ from $\vec{v}_2$ and $\vec{v}_3$. What will happen is that the span of $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ will be just the span of $\{\vec{v}_2, \vec{v}_3\}$.

We can use linearly independent vectors to create axes/coordinate systems.

Ex: Consider the vector space $\mathbb{R}^2$.

Let $\vec{i} = \langle 1, 0 \rangle$.

The vectors in the span of $\vec{i}$ create the x-axis. And we can label this the x-axis like how we label the real number line.



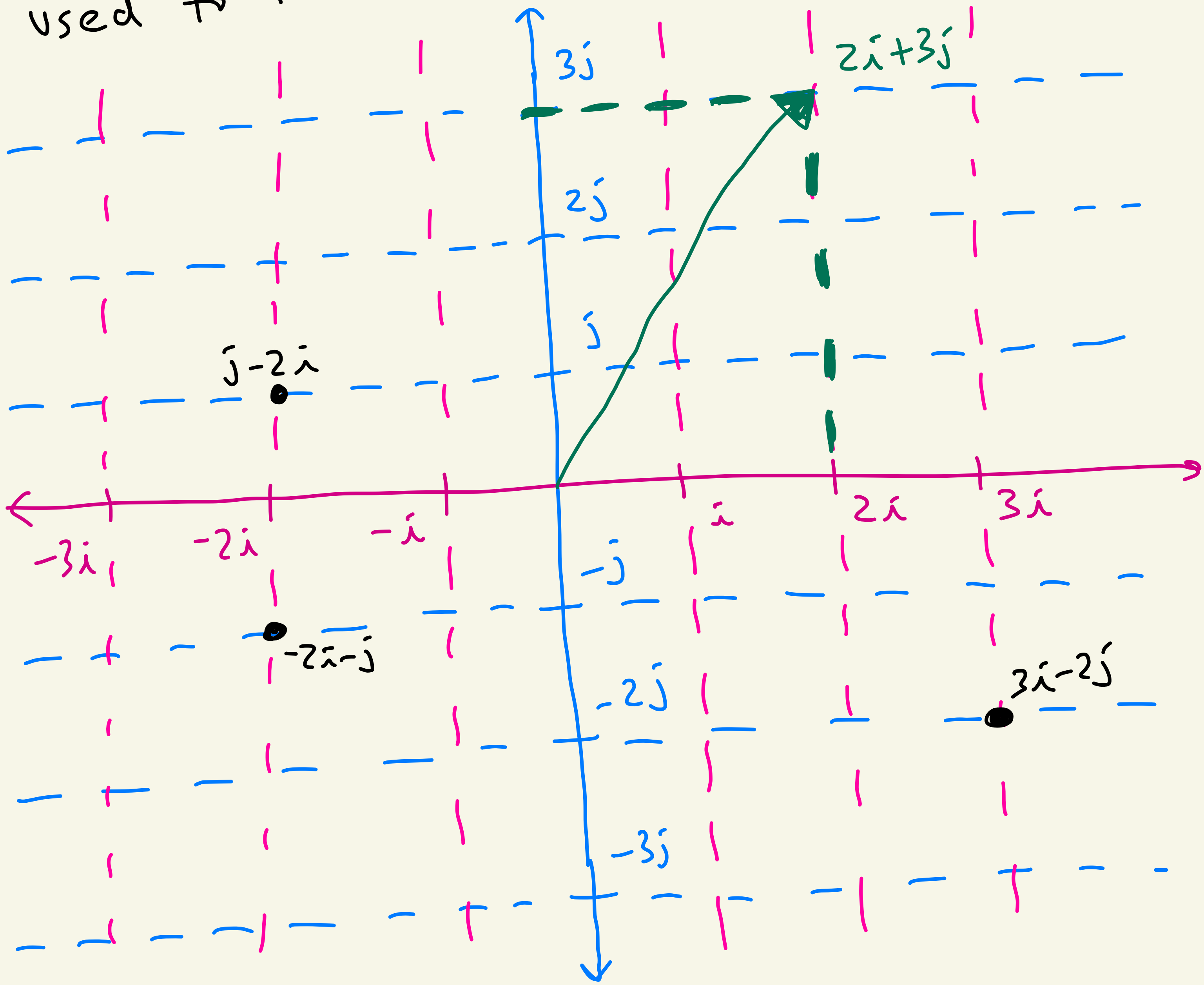
Ex: $\frac{5}{2}\vec{i}$ and $-4\vec{i}$ are drawn as examples of vectors in the span of $\vec{i}$

the vectors that live on the x-axis are the ones in the span of $\vec{i}$

Ex: Consider the vector space $\mathbb{R}^2$.

Let $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$.

These two vectors are linearly independent and create two axes that can be used to make any vector in their span.



The vector $\langle 2,3 \rangle = 2\vec{i} + 3\vec{j}$ is drawn, with the parallelogram that makes it.

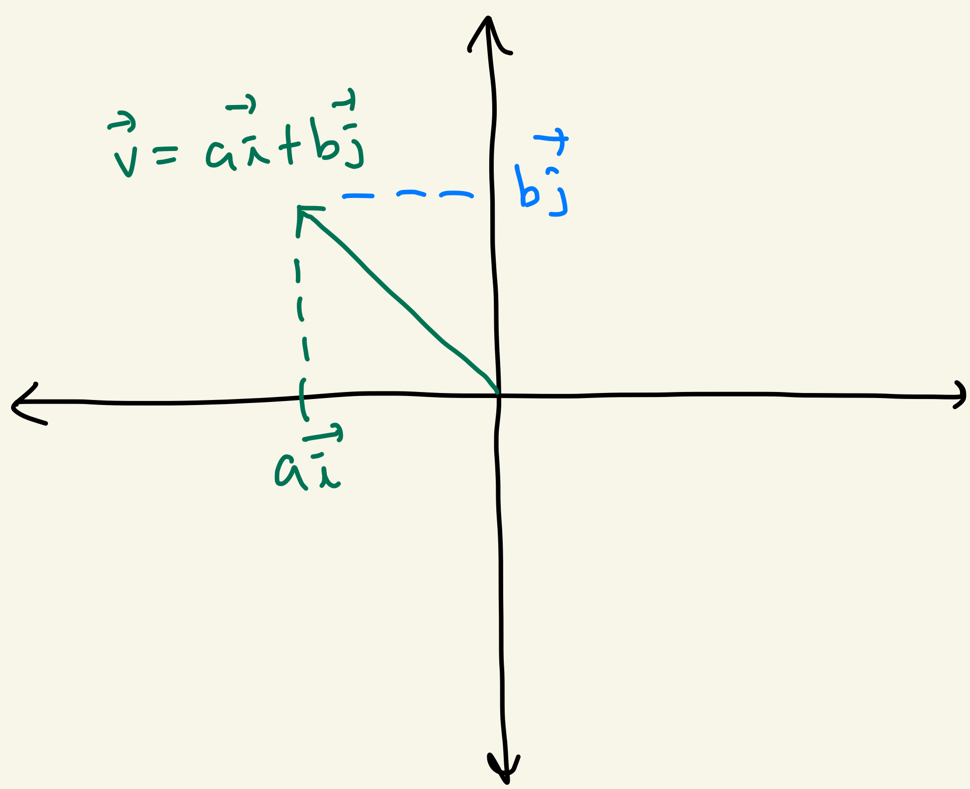
I also labeled a few other vectors.

In this example any vector $\vec{v} = \langle a, b \rangle$ can be decomposed into

$$\vec{v} = \langle a, b \rangle = \langle a, 0 \rangle + \langle 0, b \rangle$$

$$= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle$$

$$= \underbrace{a\vec{i}}_{\substack{a \\ \text{units} \\ \text{along} \\ \vec{i} \\ \text{axis}}} + \underbrace{b\vec{j}}_{\substack{b \\ \text{units} \\ \text{along} \\ \vec{j} \\ \text{axis.}}}$$



Coordinate system Theorem

Let $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be n linearly independent vectors in $\mathbb{R}^n$. Then the vectors in β create a coordinate system.

That is,

① every vector $\vec{v}$ in $\mathbb{R}^n$ is in the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ [ie you can make $\vec{v}$ by scaling and adding the $\vec{v}_i$]

② there is only one expression of the form

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

ie the $c_1, c_2, \dots, c_n$ above are unique.

Def: Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be linearly independent vectors in $\mathbb{R}^n$. We use the notation $\beta = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ to mean that we have fixed the ordering of the vectors in our set. We call β a **basis** (or **coordinate system**) for $\mathbb{R}^n$. If $\vec{v}$ is in $\mathbb{R}^n$ and

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

then we call $c_1, c_2, \dots, c_n$ the **coordinates** of $\vec{v}$ with respect to β and write

$$[\vec{v}]_\beta = \langle c_1, c_2, \dots, c_n \rangle.$$

<u>Ex:</u> Let $\beta = [\vec{i}, \vec{j}]$ in $\mathbb{R}^2$, where $\vec{i} = \langle 1,0 \rangle$ and $\vec{j} = \langle 0,1 \rangle$.

We saw earlier that β gives a linearly independent set.

Thus, since we have 2 linearly independent vectors in $\mathbb{R}^2$ we get that β is a basis for $\mathbb{R}^2$.

Let's calculate some coordinates.

Let $\vec{v} = \langle 3,1 \rangle$.

Then,
$$\vec{v} = \langle 3,1 \rangle = \langle 3,0 \rangle + \langle 0,1 \rangle$$
$$= 3\langle 1,0 \rangle + \langle 0,1 \rangle$$
$$= 3\vec{i} + \vec{j}$$

Thus, the coordinates for $\vec{v}$ with respect to β are $[\vec{v}] = \langle 3,1 \rangle$.

The basis β is called the <u>standard basis</u> for $\mathbb{R}^2$. It is the coordinate system that we usually use for the xy-plane.

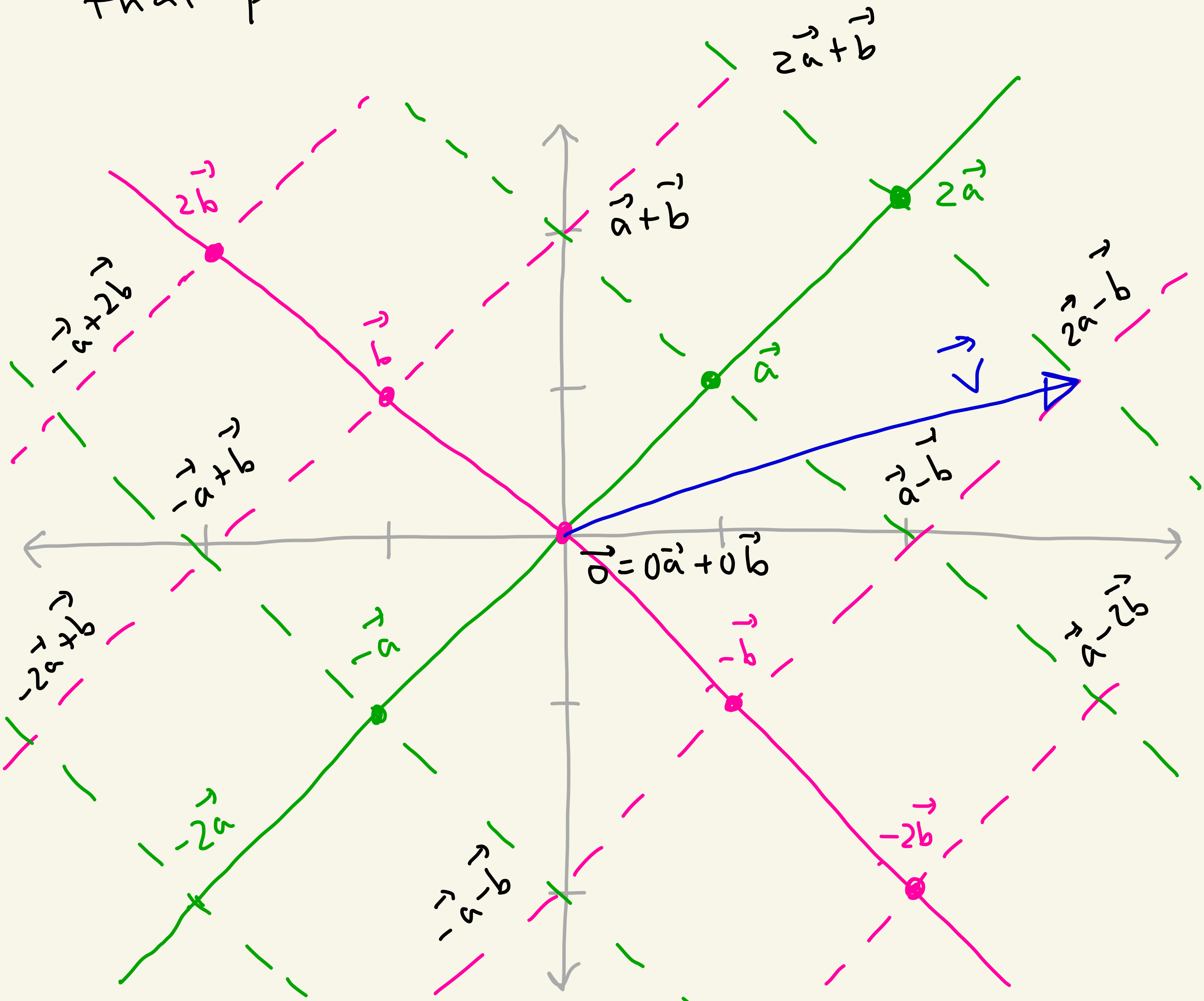
Ex: In the vector space $\mathbb{R}^2$, let $\vec{a} = \langle 1, 1 \rangle$ and $\vec{b} = \langle -1, 1 \rangle$.

We showed previously that these 2 vectors are linearly independent in $\mathbb{R}^2$.

Let $\beta = [\vec{a}, \vec{b}]$.

Then β is a basis for $\mathbb{R}^2$.

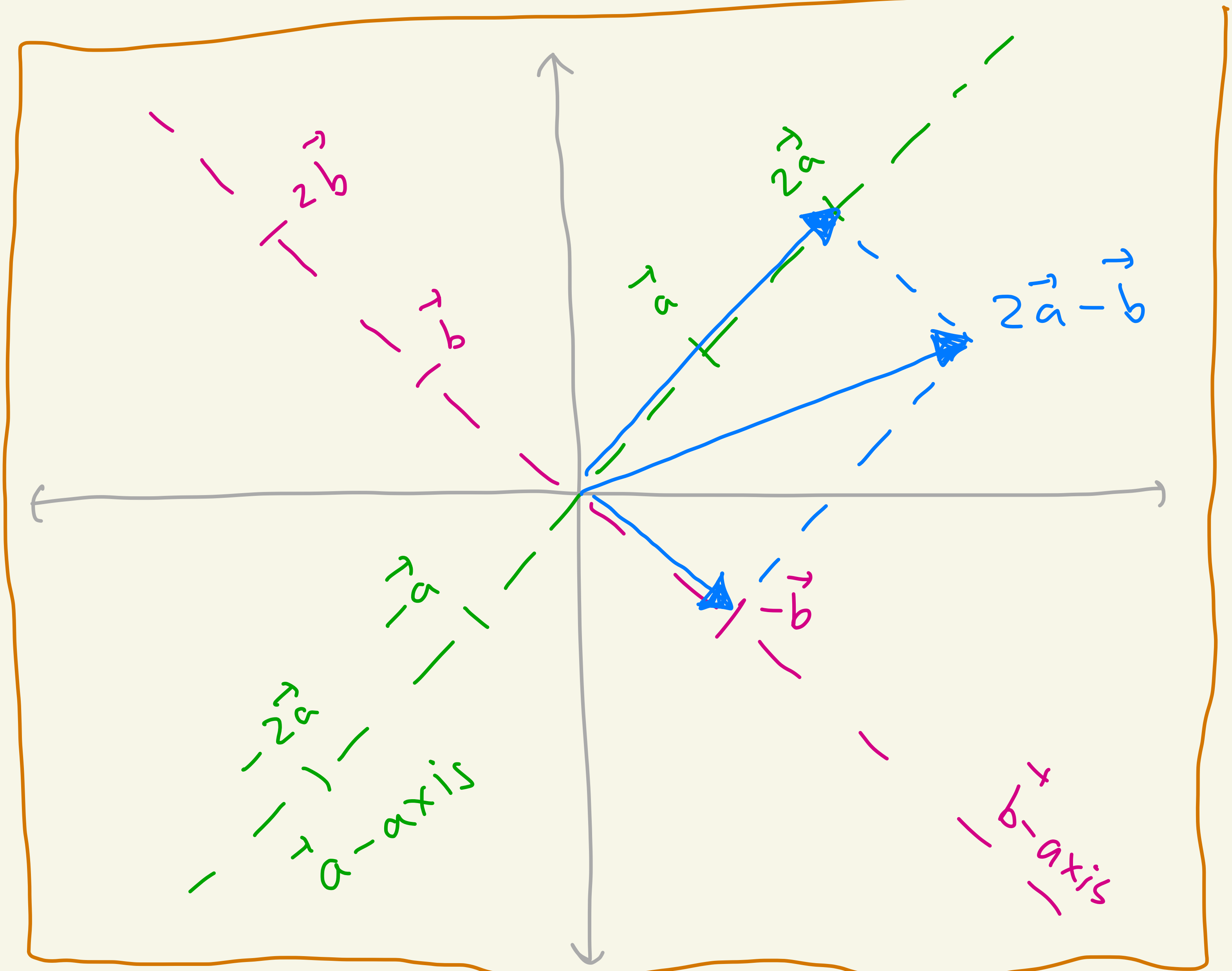
Let's draw the coordinate system that β creates.



Note: you draw the grid by drawing each line parallel to the main axis it goes with

The vector $\vec{v} = 2\vec{a} - \vec{b} = 2\langle 1,1\rangle - \langle -1,1\rangle$
$= \langle 3,1\rangle$

is drawn.



Thus, $[\vec{v}]_\beta = \langle 2,-1\rangle$ ← coordinates of $\vec{v}$ with respect to β

Ex: Let $\beta = [\vec{a}, \vec{b}]$ be as above where $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle -1,1 \rangle$.

Find the coordinates $[\vec{v}]_\beta$ for $\vec{v}$ with respect to β.

We must solve

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b}$$

for c_1, c_2.

We must solve

$$\langle 5,-1 \rangle = c_1 \langle 1,1 \rangle + c_2 \langle -1,1 \rangle$$

We get

$$\langle 5,-1 \rangle = \langle c_1 - c_2, c_1 + c_2 \rangle$$

This gives

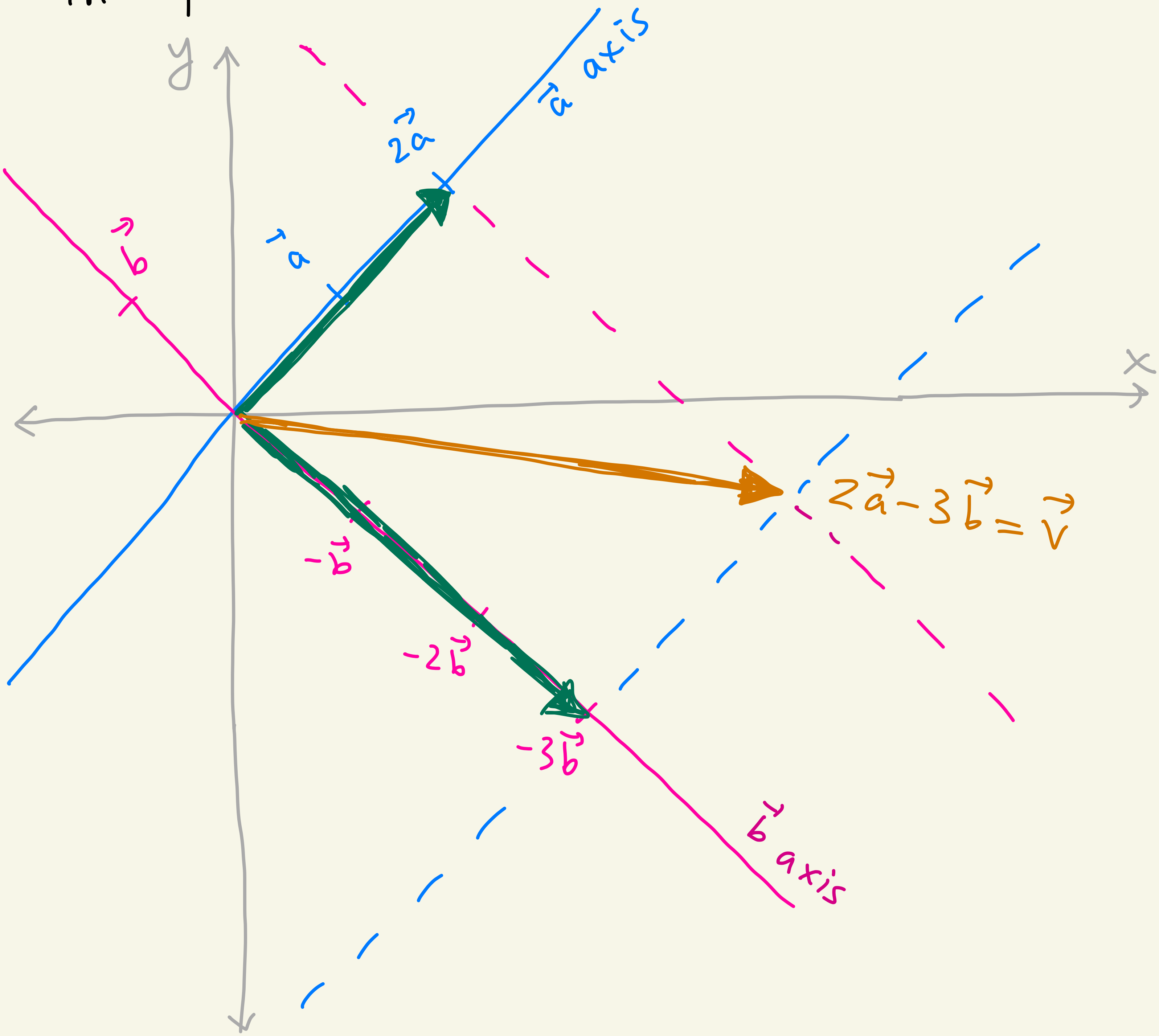
$$c_1 - c_2 = 5$$
$$c_1 + c_2 = -1$$

If you solve this system you will get $c_1 = 2$, $c_2 = -3$.

So, $\vec{v} = 2\vec{a} - 3\vec{b}$

Thus, $[\vec{v}]_\beta = \langle 2,-3 \rangle$

The picture looks like this:



Ex: Let $\beta = [\vec{a}, \vec{b}]$ as above. Suppose you know that $[\vec{v}]_\beta = \langle -2, -1 \rangle$. What is $\vec{v}$?

We know

$$\vec{v} = -2\vec{a} - 1 \cdot \vec{b} = -2\langle 1,1 \rangle - \langle -1, 1 \rangle = \langle -1, -3 \rangle$$

Thus, $\vec{v} = \langle -1, -3 \rangle$ ← note these are the standard basis coordinates for $\vec{v}$

Note:

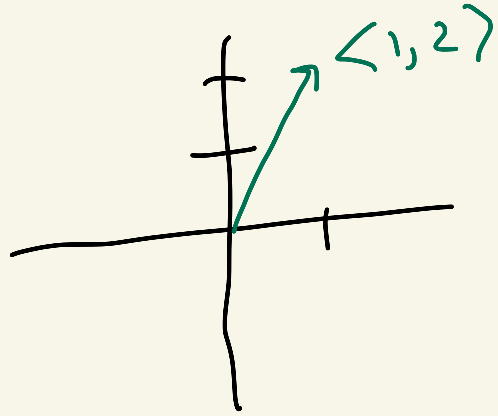
When we write $\langle x, y \rangle$ we usually mean the standard basis $\vec{i}, \vec{j}$ coordinate system.

For example

$$\langle 1, 2 \rangle$$

means

$$1 \cdot \vec{i} + 2\vec{j}$$



That's why we write $[\vec{v}]_\beta = \langle a, b \rangle$ to mean the β coordinate system.

Ex: Consider the vector space $\mathbb{R}^3$.

Let $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$

In the HW you will show that these vectors are linearly independent.

Let $\beta = [\vec{i}, \vec{j}, \vec{k}]$.

Since we have 3 linearly independent vectors in $\mathbb{R}^3$ we get that β is a basis for $\mathbb{R}^3$.

Given a vector $\vec{v} = \langle x,y,z\rangle$ we can write

$$\begin{aligned}
\vec{v} &= \langle x,y,z\rangle \\
&= \langle x,0,0\rangle + \langle 0,y,0\rangle + \langle 0,0,z\rangle \\
&= x\langle 1,0,0\rangle + y\langle 0,1,0\rangle + z\langle 0,0,1\rangle \\
&= x\vec{i} + y\vec{j} + z\vec{k}
\end{aligned}$$

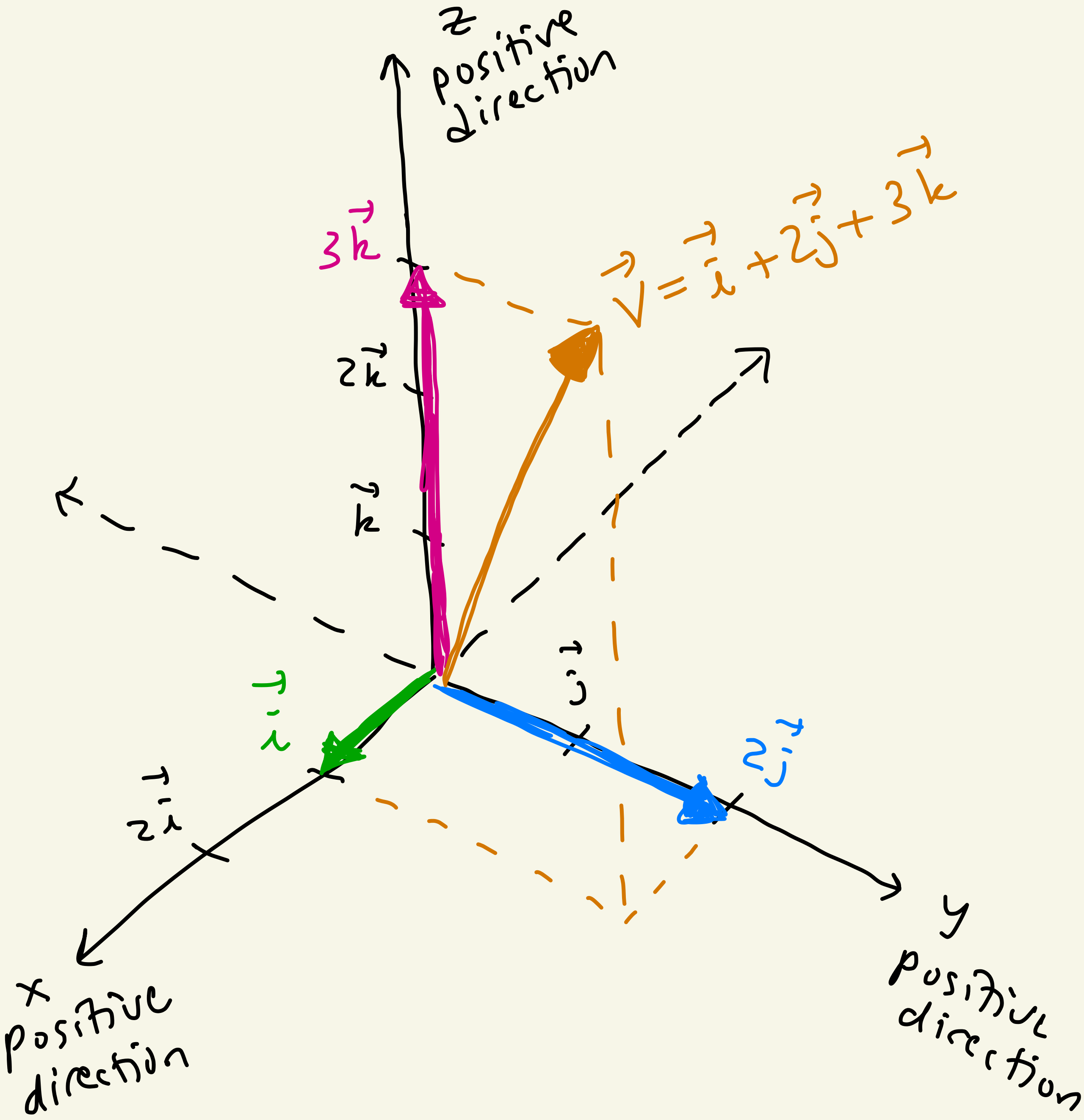
So, $[\vec{v}]_\beta = \langle x,y,z\rangle$.

This is how you can decompose $\vec{v}$ into its $\vec{i}, \vec{j}, \vec{k}$ coordinates

For example, let $\vec{v} = \langle 1, 2, 3 \rangle$

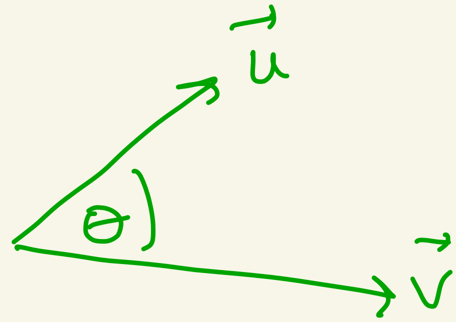
Then, $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$

So, $[\vec{v}]_\beta = \langle 1, 2, 3 \rangle$



$\beta = [\vec{i}, \vec{j}, \vec{k}]$ is called the <u>standard basis</u> for $\mathbb{R}^3$.

Recall that if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ that



$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$$

where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$.

Therefore, $\theta = 90^\circ$ exactly when

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \underbrace{\cos(90^\circ)}_{\cos(\frac{\pi}{2}) = 0} = 0$$

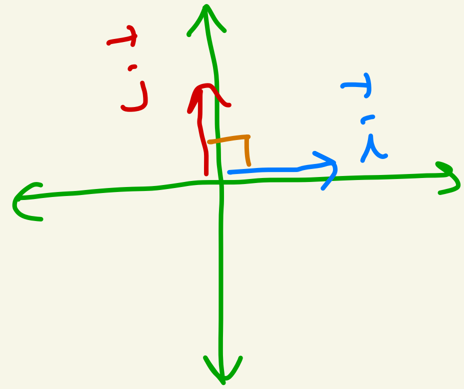
Using $\mathbb{R}^2$ and $\mathbb{R}^3$ as intuition we extend this idea to any dimension n.

<u>Def:</u>

We say that two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are <u>orthogonal</u> if $\vec{u} \cdot \vec{v} = 0$.

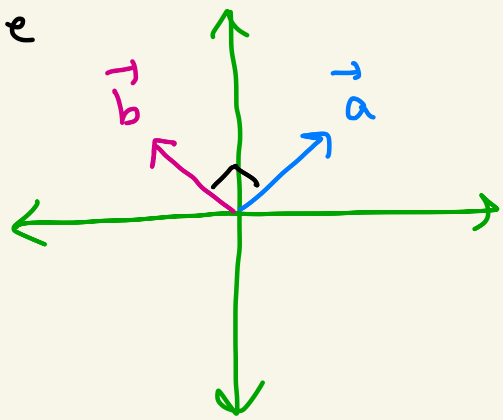
Ex: $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ are orthogonal since

$$\vec{i} \cdot \vec{j} = 1 \cdot 0 + 0 \cdot 1 = 0$$



Ex: $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$ are orthogonal since

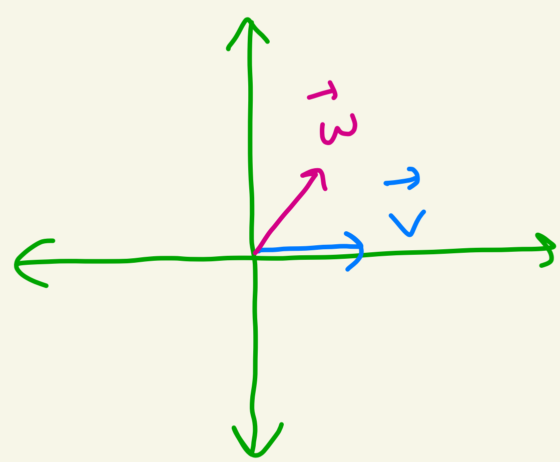
$$\vec{a} \cdot \vec{b} = (1)(-1) + (1)(1) = 0$$



Ex: Let $\vec{v} = \langle 1, 0 \rangle$, $\vec{w} = \langle 1, 1 \rangle$.

Then, $\vec{v} \cdot \vec{w} = (1)(1) + (0)(1) = 1$

Since $\vec{v} \cdot \vec{w} \neq 0$ the vectors are not orthogonal.



<u>Def:</u> Let $\beta = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ be a basis for $\mathbb{R}^n$.

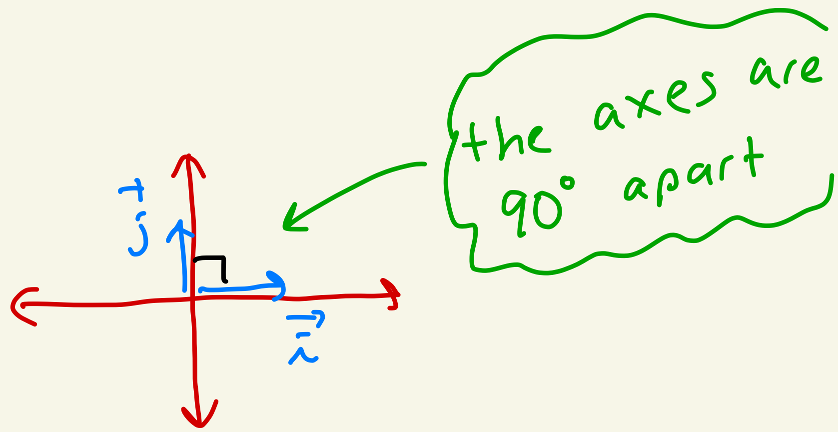
- We say that β is an <u>orthogonal</u> basis if every pair of vectors from the basis are orthogonal to each other, that is if $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$.
- We say that β is an <u>orthonormal</u> basis if β is an orthogonal basis and every vector in β has length 1.

<u>Ex:</u> In the vector space $\mathbb{R}^2$

let $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$.

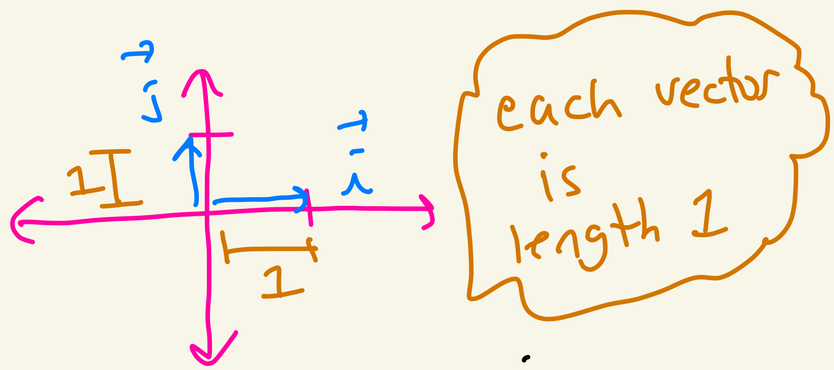
Let $\beta = [\vec{i}, \vec{j}]$.

① We know that β is a basis for $\mathbb{R}^2$.



② $\vec{i} \cdot \vec{j} = 0$

So, $\vec{i}, \vec{j}$ form an orthogonal basis.



③ $\|\vec{i}\| = \sqrt{1^2 + 0^2} = 1$

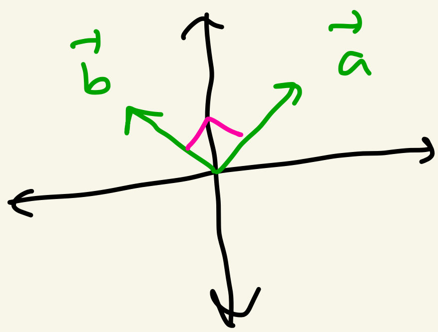
$\|\vec{j}\| = \sqrt{0^2 + 1^2} = 1$

Thus, $\vec{i}, \vec{j}$ form an orthonormal basis

<u>Result:</u> $\beta = [\vec{i}, \vec{j}]$ is an orthonormal basis.

Ex: In the vector space $\mathbb{R}^2$,
let $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$.
Let $\beta = [\vec{a}, \vec{b}]$.

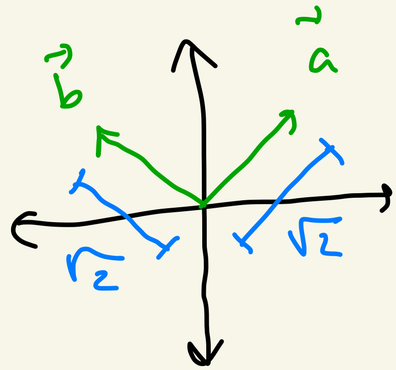
① We saw earlier that β is a basis for $\mathbb{R}^2$.



the axes are 90° apart

② $\vec{a} \cdot \vec{b} = 0$

Thus, $\vec{a}$ and $\vec{b}$ form an orthogonal basis



③ $\|\vec{a}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\|\vec{b}\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

The lengths of the vectors are not 1 thus $\vec{a}, \vec{b}$ are not an orthonormal basis

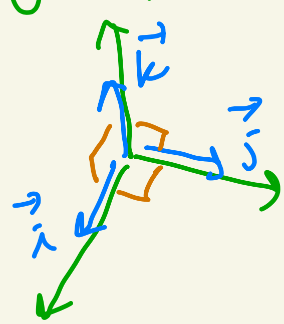
Result: β is an orthogonal basis but not an orthonormal basis for $\mathbb{R}^2$.

<u>Ex:</u> In the vector space $\mathbb{R}^3$, let $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$.

Let $\beta = [\vec{i}, \vec{j}, \vec{k}]$.

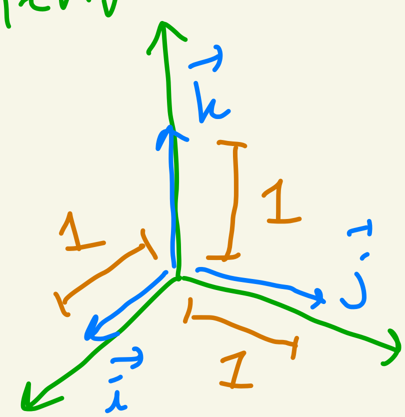
① In HW you show that β is a basis for $\mathbb{R}^3$.

② $$\left.\begin{aligned} \vec{i}\cdot\vec{j} &= 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0 \\ \vec{i}\cdot\vec{k} &= 1\cdot 0 + 0\cdot 0 + 0\cdot 1 = 0 \\ \vec{j}\cdot\vec{k} &= 0\cdot 0 + 1\cdot 0 + 0\cdot 0 = 0 \end{aligned}\right\}$$ each axis is 90° apart



Thus, $\vec{i}, \vec{j}, \vec{k}$ form an orthogonal basis for $\mathbb{R}^3$

③ $$\left.\begin{aligned} \|\vec{i}\| &= \sqrt{1^2 + 0^2 + 0^2} = 1 \\ \|\vec{j}\| &= \sqrt{0^2 + 1^2 + 0^2} = 1 \\ \|\vec{k}\| &= \sqrt{0^2 + 0^2 + 1^2} = 1 \end{aligned}\right\}$$ each vector is length 1



Result: β is an orthonormal basis for $\mathbb{R}^3$.

Ex: In the vector space $\mathbb{R}^3$,
let $\vec{a} = \langle 1,0,0 \rangle$, $\vec{b} = \langle 1,1,0 \rangle$, $\vec{c} = \langle 1,1,1 \rangle$

① Let's show $\vec{a}, \vec{b}, \vec{c}$ are lin. ind.

Consider
$$c_1 \vec{a} + c_2 \vec{b} + c_3 \vec{c} = \vec{0}$$

We get
$$c_1 \langle 1,0,0 \rangle + c_2 \langle 1,1,0 \rangle + c_3 \langle 1,1,1 \rangle = \langle 0,0,0 \rangle$$
$$\langle c_1 + c_2 + c_3,\ c_2 + c_3,\ c_3 \rangle = \langle 0,0,0 \rangle$$

This gives
$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}} \quad \begin{aligned} &\text{(i)} \\ &\text{(ii)} \\ &\text{(iii)} \end{aligned}$$

This system is already reduced.
Back-substituting gives:

(i) $c_3 = 0$
(ii) $c_2 = -c_3 = -(0) = 0$
(iii) $c_1 = -c_2 - c_3 = -(0) - (0) = 0$

Thus, the only solution to

$$c_1\vec{a} + c_2\vec{b} + c_3\vec{c} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

So, $\vec{a}, \vec{b}, \vec{c}$ are lin. ind.

Let $\beta = [\vec{a}, \vec{b}, \vec{c}]$.

Since we have 3 linearly independent vectors in $\mathbb{R}^3$ we have that β is a basis for $\mathbb{R}^3$.

(2) We have that

$$\left.\begin{aligned} \vec{a}\cdot\vec{b} &= 1 \\ \vec{b}\cdot\vec{c} &= 2 \\ \vec{a}\cdot\vec{c} &= 1 \end{aligned}\right\}$$ the axes created are not at $90°$ to each other

We do not have an orthogonal basis here.

We don't check orthonormal since that requires at least orthogonal.

Thus, $\beta = [\vec{a}, \vec{b}, \vec{c}]$ is a basis for $\mathbb{R}^3$ that isn't orthogonal or orthonormal.

Let's decompose a vector into this coordinate system.

Let $\vec{v} = \langle 4, 2, 3 \rangle$.

Let's solve

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b} + c_3 \vec{c}$$

for c_1, c_2, c_3.

We get

$$\langle 4,2,3 \rangle = c_1 \langle 1,0,0 \rangle + c_2 \langle 1,1,0 \rangle + c_3 \langle 1,1,1 \rangle$$

which gives

$$\langle 4,2,3 \rangle = \langle c_1 + c_2 + c_3,\ c_2 + c_3,\ c_3 \rangle$$

or

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_2 + c_3 &= 2 \\ c_3 &= 3 \end{aligned}} \quad \begin{aligned} &(i) \\ &(ii) \\ &(iii) \end{aligned}$$

This system is already reduced so we get

We get

(iii) $c_3 = 3$

(ii) $c_2 = 2 - c_3 = 2 - 3 = -1$

(i) $c_1 = 4 - c_2 - c_3 = 4 - (-1) - 3 = 2$

Thus, $\boxed{\vec{v} = 2\vec{a} - \vec{b} + 3\vec{c}}$. So, $[\vec{v}]_\beta = \langle 2, -1, 3 \rangle$

Coordinate dot-product theorem

Let $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be an orthogonal basis (coordinate system) in $\mathbb{R}^n$.

Let $\vec{v}$ be any vector in $\mathbb{R}^n$.

Then,

$$\vec{v} = \underbrace{\boxed{\frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}}}_{c_1}\vec{v}_1 + \underbrace{\boxed{\frac{\vec{v}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}}}_{c_2}\vec{v}_2 + \cdots + \underbrace{\boxed{\frac{\vec{v}\cdot\vec{v}_n}{\|\vec{v}_n\|^2}}}_{c_n}\vec{v}_n$$

Moreover if β is an orthonormal basis (ie, each of $\vec{v}_i$ has length 1) then

$$\vec{v} = \underbrace{\boxed{(\vec{v}\cdot\vec{v}_1)}}_{c_1}\vec{v}_1 + \underbrace{\boxed{(\vec{v}\cdot\vec{v}_2)}}_{c_2}\vec{v}_2 + \cdots + \underbrace{\boxed{(\vec{v}\cdot\vec{v}_n)}}_{c_n}\vec{v}_n$$

Note: This theorem only works for orthogonal or orthonormal coordinate systems

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{i}, \vec{j}]$

we saw that β is an orthonormal basis for $\mathbb{R}^2$.

Let's decompose $\vec{v} = \langle 3, -6 \rangle$ in terms of this basis.

We want to solve

$$\vec{v} = c_1 \vec{i} + c_2 \vec{j}$$

The coordinate dot product theorem tells us that

$$c_1 = \vec{v} \cdot \vec{i} = \langle 3, -6 \rangle \cdot \langle 1, 0 \rangle = 3 \cdot 1 - 6 \cdot 0 = 3$$

$$c_2 = \vec{v} \cdot \vec{j} = \langle 3, -6 \rangle \cdot \langle 0, 1 \rangle = 3 \cdot 0 - 6 \cdot 1 = -6$$

Thus,

$$\vec{v} = 3\vec{i} - 6\vec{j}$$

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{a}, \vec{b}]$ where

$\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle -1,1 \rangle$.

We saw that β is an orthogonal basis.

Let $\vec{v} = \langle 5,-1 \rangle$.

Let's find $\vec{v}$'s β-coordinates.

We want to solve

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b}$$

By the coordinate dot product theorem

$$c_1 = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} = \frac{\langle 5,-1 \rangle \cdot \langle 1,1 \rangle}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{5-1}{2} = 2$$

and

$$c_2 = \frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2} = \frac{\langle 5,-1 \rangle \cdot \langle -1,1 \rangle}{\left(\sqrt{(-1)^2+1^2}\right)^2} = \frac{-5-1}{2} = -3$$

Thus,

$$\vec{v} = 2\vec{a} - 3\vec{b}$$

Check:

$$2\vec{a} - 3\vec{b} = 2\langle 1,1 \rangle - 3\langle -1,1 \rangle = \langle 5,-1 \rangle = \vec{v}$$

$\boxed{\text{How to turn an orthogonal basis into an orthonormal basis}}$

Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$.

Then,

$$\beta' = \left\{ \frac{1}{\|\vec{v}_1\|}\vec{v}_1, \frac{1}{\|\vec{v}_2\|}\vec{v}_2, \ldots, \frac{1}{\|\vec{v}_n\|}\vec{v}_n \right\}$$

divide each vector by its length

will be an orthonormal basis

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{a}, \vec{b}]$

where $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$.

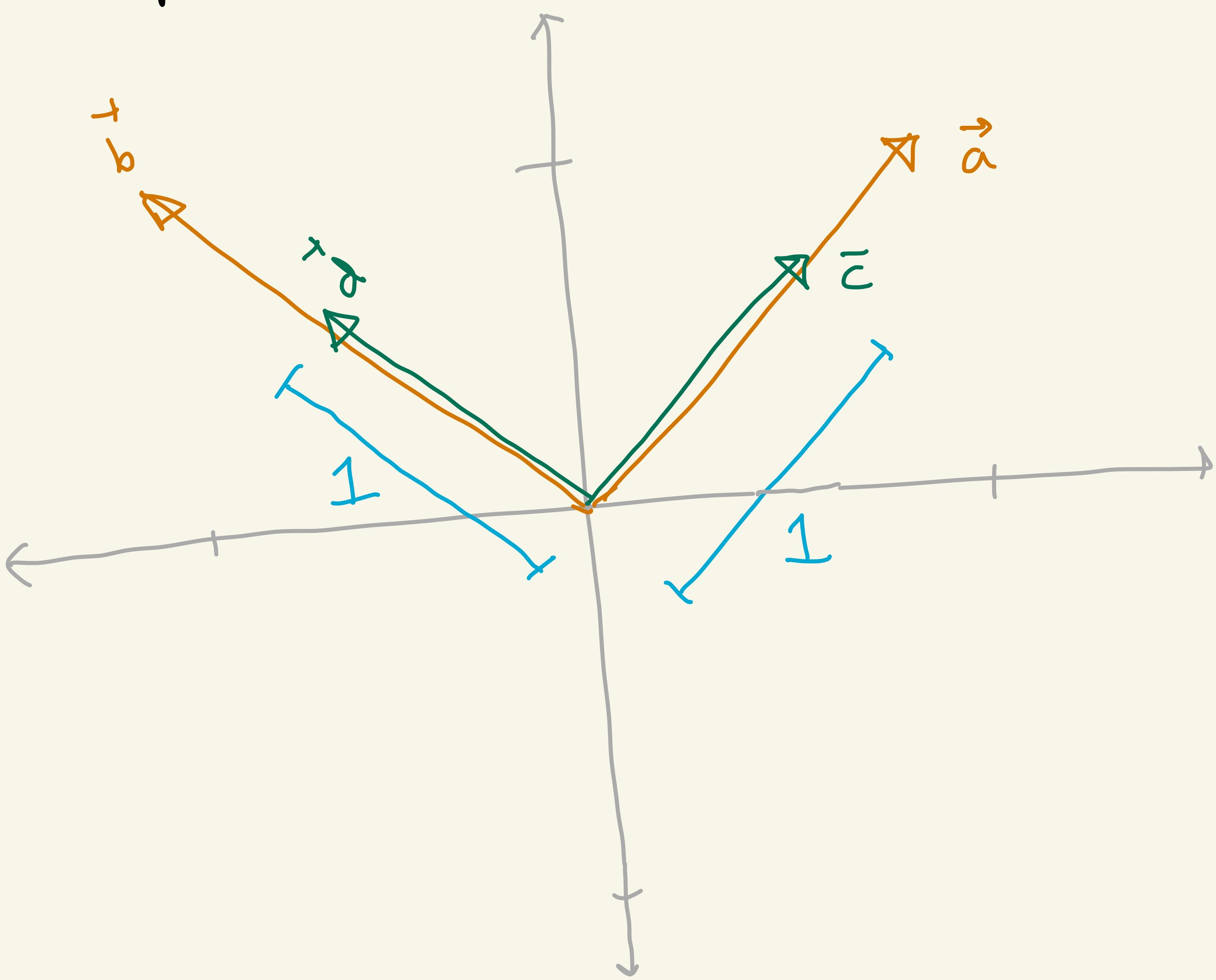
We saw that β is an orthogonal basis.

Let

$$\vec{c} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{2}}\langle 1,1\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{d} = \frac{1}{\|\vec{b}\|}\vec{b} = \frac{1}{\sqrt{2}}\langle -1,1\rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Then $\beta' = [\vec{c}, \vec{d}]$ is an orthonormal basis.



Let's decompose $\vec{v} = \langle 5, -1 \rangle$ in terms of this new basis β'.

Recall:

$\vec{c} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\vec{d} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

We want to solve

$$\vec{v} = c_1 \vec{c} + c_2 \vec{d}$$

By the coordinate dot product theorem we get

$$c_1 = \vec{v} \cdot \vec{c} = \langle 5, -1 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$c_2 = \vec{v} \cdot \vec{d} = \langle 5, -1 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{-6}{\sqrt{2}} = -3\sqrt{2}$$

So,

$$\vec{v} = 2\sqrt{2}\, \vec{c} - 3\sqrt{2}\, \vec{d}.$$

Check:

$$2\sqrt{2}\, \vec{c} - 3\sqrt{2}\, \vec{d} = 2\sqrt{2} \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle - 3\sqrt{2} \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$= \langle 2, 2 \rangle + \langle 3, -3 \rangle$$

$$= \langle 5, -1 \rangle = \vec{v}$$

Thus, $[\vec{v}]_{\beta'} = [2\sqrt{2}, -3\sqrt{2}]$.

Recall before that $[\vec{v}]_\beta = \langle 2, -3 \rangle$

Def: The standard basis for $\mathbb{R}^n$ is $\beta = [\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n]$ where $\vec{e}_i$ has a 1 in spot i and 0's everywhere else.

$\mathbb{R}^n$	standard basis
$\mathbb{R}^2$	$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
$\mathbb{R}^3$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$
$\mathbb{R}^4$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$
$\vdots$	$\vdots$

Note: When we write a vector $\vec{v}$ in $\mathbb{R}^n$ we always write the standard basis coordinates. To change coordinates we write $[\vec{v}]_\beta$.

For example, in $\mathbb{R}^3$, $\vec{v} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ means

$$\vec{v} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

But if $\beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$ then

$\underbrace{[\vec{v}]_\beta = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}}_{\beta \text{ coordinates}}$ means that

$$\vec{v} = \underbrace{3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{converting back to standard coordinates}} = \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$$